Eigenvalues of infinite non-compact star graph

Gökhan Mutlu

Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey
E-mail: gmutlu@gazi.edu.tr

Abstract

We consider the Schrdinger operator on an infinite non-compact star graph which has countably infinite number of semi-infinite rays emanating from the single vertex of the graph. We impose the most general vertex conditions at the central vertex. We transfer this boundary value problem on the infinite non-compact quantum star graph to Schrdinger equation on the half-line with operator coefficients in an infinite dimensional separable Hilbert space. We obtain the characteristic function of this boundary value problem which is the Jost function of the Schrdinger equation on the half-line with operator coefficients.

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1 Introduction

Differential operators on metric graphs have been studied extensively in last two decades due to their numerous applications in modeling problems in mathematics, physics, chemistry, and engineering. These models include for example; free-electron theory of conjugated molecules, quantum chaos, quantum wires, dynamical systems, photonic crystals, scattering theory etc (see survey [3] for more details about more applications and models). In 1997, Kottos and Smilansky [4] proposed differential operators on metric graphs as a model to study quantum chaos and then the term quantum graph has been widely used thereafter.

A metric graph is obtained by assigning a positive length L_e to each edge $e \in E$ of a graph $G = (V, E)$ with a vertex set V and edge set E. By doing so, we can identify each edge e of a metric graph with an interval $[0, L_e]$ of the real line. This identification gives rise to a local coordinate $x_e \in [0, L_e]$ of each point on the edge $e \in E$ and also a natural direction to the edge starting from $x_e = 0$ and ending at $x_e = L_e$. Now we can consider a metric graph as a collection of intervals where endpoints of edges corresponding to the same vertex are identified. In a connected metric graph one can define a natural metric to turn the graph into a metric space which explains the name (see [5] for more details). If the sets V and E are finite and every edge has a finite length then the metric graph $G = (V, E)$ is called a compact graph. A compact metric graph is also compact as a topological space. Note that the points of a metric graph are not just the vertices as in discrete or combinatorial graphs but all intermediate points on the edges as well. One can naturally carry the Lebesgue measure on the real line to a metric graph and also define the standard function spaces as a direct sum of corresponding function spaces on individual edges.

On a metric graph, we can define a differential operator (Hamiltonian) acting on the edges of the graph with appropriate boundary conditions imposed at vertices. A quantum graph is a collection of a metric graph, a Hamiltonian acting on it and vertex conditions given for each vertex. In the literature, most of the studies concern the Laplace operator $f \to -\frac{d}{dx^2} f$ (see [5]) or a more general

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Schrödinger operator $f \to -\frac{d}{dx^2}f + V(x)f$ with decent potentials $V(x)$ (see [6, 7, 8]). As for vertex conditions, the most common vertex conditions are Neumann or Neumann-Kirchoff conditions given by

$$f_i(v) = f_j(v), \text{for all edges } e \text{ such that } v \in e_i \text{ (continuity condition)},$$

$$\sum_{v \in e} \frac{df}{dx_e}(v) = 0 \text{ (current conservation condition)},$$

where $v \in e$ means that the vertex v is incident in edge e and $f_i = f_{|e_i}$, and the derivatives are taken in outgoing directions (from vertex to the edge). There are other types of vertex conditions such as $\delta-$type conditions, extended $\delta-$type conditions, vertex Dirichlet condition ($f(v) = 0$) etc (see [5]). In general, vertex conditions at a vertex v with degree d can be written in matrix form

$$A_v F(v) + B_v F'(v) = 0,$$

where A_v and B_v are $d \times d$ matrices, $F(v)$ and $F'(v)$ denote the vector of function values and its outgoing derivatives at v, respectively.

Spectral properties of compact quantum graphs are well known. It has been proven that if $rank\,(A_v\ B_v)$ is maximal and $A_v B_v^* = B_v A_v^*$ for each vertex v of a compact quantum graph, then the quantum graph is selfadjoint when considering Schrödinger operator as a Hamiltonian [11]. In particular, the spectrum consists of isolated eigenvalues with finite multiplicities [5]. First studies about non-compact graphs were conducted in [9, 10].

Star graphs are the simplest non-trivial graphs to study. Moreover, every graph locally (around a vertex) looks like a star graph. For these reasons, studying star graphs is crucial. For a complete treatment of differential operators acting on star graphs we refer the reader to [12]. Most of the studies in the literature consider the compact or finite star graphs. However, in this study, we consider an infinite and non-compact star graph. Namely, we consider a star graph with a single vertex and countably infinite number of semi-infinite rays with infinite length attached at the vertex. We consider the Schrödinger operator with complex-valued potentials as Hamiltonian and impose the most general vertex conditions at the central vertex. We transfer this boundary value problem to Schrödinger equation on the half-line with operator coefficients in a separable infinite dimensional Hilbert space. Spectral properties of such equations with selfadjoint operator coefficients have been studied well [2, 13, 14, 15]. We obtain the characteristic function of this boundary value problem which coincides with the Jost function of Schrödinger equation with infinite dimensional operator coefficients. Finally, we present some examples including the Dirichlet boundary conditions and Robin conditions.

2 Statement of the problem and some auxiliary results

We consider a star graph Γ with a single vertex v and countably infinite number of rays e_j $(j = 1, 2, ...)$ with infinite length emanating from the vertex. In this metric graph, each ray is identified with the interval $[0, \infty)$ and the origin of each ray is identified with the single vertex of the graph. We consider the Schrödinger operator

$$H : y_j \to -y_j^{''} + q_j(x_j) y_j \tag{2.1}$$

on the edges e_j $(j = 1, 2, ...)$ of the graph where the potentials $q_j(x)$ $(j = 1, 2, ...)$ are complex-valued measurable functions on $[0, \infty)$ which have a finite first moment i.e.,

$$\int_0^\infty (1 + x) |q_j(x)| \, dx < \infty.$$

We consider the most general vertex conditions at the single vertex of the graph given by

$$AF(v) + BF^{'}(v) = 0,$$

where A and B are linear operators in an infinite dimensional, separable, complex Hilbert space Ω and $F(v)$ and $F^{'}(v)$ denote the infinite dimensional vector of values of the function and its outgoing (from vertex to the edge) first derivative at the vertex v, respectively. Parameterization of the rays in the direction starting from the vertex yields the representation of the vertex conditions in the following form

$$AF(0) + BF^{'}(0) = 0, \tag{2.2}$$

where $F(0) = (y_1(0), y_2(0), ...)^T$ and $F^{'}(0) = \left(y_1^{'}(0), y_2^{'}(0), ...\right)^T$.

A function on Γ is a collection of functions on its edges. Therefore, a function f on Γ is a vector function defined on $[0, \infty)$. The Hilbert space $L_2(\Gamma)$ of functions defined on Γ consists of functions f that are measurable and square integrable on each edge e_j and such that

$$\|f\|^2_{L_2(\Gamma)} := \sum_{n=1}^{\infty} \|f_n\|^2_{L_2(0,\infty)} < \infty.$$

The domain of the Hamiltonian on Γ which is defined by (2.1) and subject to the vertex conditions (2.2) consists of functions $f \in L_2(\Gamma)$ such that

- $f_j \in H^2(0, \infty)$ for each $j = 1, 2, ...$
- f satisfies the vertex conditions (2.2).

Let us denote by $H_1 := L_2((0, \infty), \Omega)$ the Hilbert space of vector-valued functions $f : [0, \infty) \to \Omega$ such that f is Bochner-integrable in each finite subinterval of $[0, \infty)$ and that

$$\|f\|^2_{H_1} := \int_0^\infty \|f(x)\|^2_\Omega \, dx < \infty.$$

Consider the Schrödinger operator L defined on H_1

$$L : f \to -f^{''} + Q(x)f$$

where $Q(x) = diag(q_j(x))_{j=1}^\infty$ is a diagonal operator in Ω for each $x \in [0, \infty)$ and the boundary condition at the origin is given by

$$Af(0) + Bf^{'}(0) = 0, \tag{2.3}$$

where A and B are linear operators appearing in (2.2).

Obviously the Hamiltonian on Γ is equivalent in terms of spectral analysis to the Schrödinger operator L defined on H_1. Therefore, let us consider the Schrödinger's operator equation on the half-line

$$-y^{''}+Q(x)y=k^2y, \tag{2.4}$$

where $Q(x)=diag(q_j(x))_{j=1}^{\infty}$ is a diagonal operator in Ω for each $x\in[0,\infty)$, $y\in H_1$ and k^2 is a spectral parameter. Note that in equation (2.4) we can also consider y as an operator function (i.e., $y(x)$ is a linear operator in Ω for each $x\in[0,\infty)$). It is well known [1] that there is a one to one correspondence between the sequences (y_n) of vector solutions and operator solutions of equation (2.4).

It is well known that there exists an operator solution of equation (2.4) called the Jost solution $F(k,x)$ which satisfies the following asymptotic relations for $k\in\overline{\mathbb{C}_+}:=\{z\in\mathbb{C}:\ Imz\geq 0\}$

$$F(k,x)=e^{ikx}\left[I+\circ(1)\right],\ F_x(k,x)=ike^{ikx}\left[I+\circ(1)\right],\ x\to\infty, \tag{2.5}$$

where I denotes the identity operator in Ω. It is also known that $F(k,x)$ and $F_x(k,x)$ are analytic in $k\in\mathbb{C}_+:=\{z\in\mathbb{C}:\ Imz>0\}$ and continuous including the real axis for each fixed x [1]. It is clear that $F(k,x)$ is a diagonal operator in Ω for each $x\in[0,\infty)$ where each nonzero entry is the Jost solution of the equation $-y_j^{''}+q_j(x)y_j=k^2y_j$ on the corresponding edge e_j. From (2.5) it easily follows that each nonzero entry of $F(k,x)$ decays exponentially to zero as $x\to\infty$ for each fixed $k\in\mathbb{C}_+$.

Equation (2.4) has another operator solution $G(k,x)$ which satisfies [1] the asymptotic relations for $k\in\overline{\mathbb{C}_+}$

$$G(k,x)=e^{-ikx}\left[I+\circ(1)\right],\ G_x(k,x)=-ike^{-ikx}\left[I+\circ(1)\right],\ x\to\infty. \tag{2.6}$$

$G(k,x)$ and $G_x(k,x)$ are also analytic in $k\in\mathbb{C}_+$ and continuous including the real axis for each fixed x [1]. From (2.6) it easily follows that each nonzero entry of $G(k,x)$ grows exponentially as $x\to\infty$ for each fixed $k\in\mathbb{C}_+$.

It is well known that there exist operator solutions $C(k,x),\ S(k,x)$ of equation (2.4) which satisfy the initial conditions [1]

$$C(k,0)=S^{'}(k,0)=I,\ S(k,0)=C^{'}(k,0)=0.$$

It is also known that there exist operator solutions of equation (2.4) called the regular solution that can be specified via constant initial conditions at a finite x value. These solutions are entire in k for each fixed x. In particular, there exists an operator solution $\varphi(k,x)$ of equation (2.4) which satisfies the initial conditions

$$\varphi(k,0)=B,\ \varphi^{'}(k,0)=-A, \tag{2.7}$$

where A and B are linear operators appearing in (2.2). In fact

$$\varphi(k,x)=-S(k,x)A+C(k,x)B.$$

Let $Y(k,x),\ Z(k,x)$ be operator solutions of (2.4). Let us define the Wronskian of these solutions as

$$W\left[Y,Z\right](x):=Y(x)Z^{'}(x)-Y^{'}(x)Z(x).$$

Since $Y(k,x)$ and $Z(k,x)$ are diagonal operators it easily follows that $W\left[Y,Z\right](x)$ is independent of the variable x.

Let us define the Jost operator $J(k)$ for $k\in\overline{\mathbb{C}_+}\setminus\{0\}$ as the Wronskian of the Jost solution and regular solution

$$J(k):=W\left[\varphi,F\right](x)=\varphi(k,x)F^{'}(k,x)-\varphi^{'}(k,x)F(k,x)$$

Since this Wronskian is independent of x, evaluating at $x=0$ and (2.7) yield

$$\begin{aligned} J(k) &= \varphi(k,0)F^{'}(k,0)-\varphi^{'}(k,0)F(k,0) \\ &= AF(k,0)+BF^{'}(k,0) \end{aligned} \tag{2.8}$$

3 Main results

Theorem 3.1. k^2 is an eigenvalue of the infinite star graph Γ (or equivalently Schrödinger operator L defined on H_1) if and only if the Jost operator $J(k)$ is not invertible for $k\in\mathbb{C}_+$.

Proof. It is obvious that the eigenvalues of the infinite star graph Γ and Schrödinger operator L defined on H_1 coincide. Suppose k^2 is an eigenvalue of Schrödinger operator L defined on H_1. It is well known that (see [1]) for each $k\in\overline{\mathbb{C}_+}\setminus\{0\}$, every vector solution $y(k,x)$ of (2.4) can be written

$$y(k,x)=F(k,x)\alpha+G(k,x)\beta,$$

for some constant vectors $\alpha,\beta\in\Omega$. In order to have both $F(k,x)\alpha, G(k,x)\beta\in H_1$ it follows that $Imk\neq 0$. Moreover, $G(k,x)\beta\notin H_1$ for $k\in\mathbb{C}_+$. Therefore, we must have $y(k,x)=F(k,x)\alpha$. Imposing the boundary condition (2.3)

$$\left[AF(k,0)+BF^{'}(k,0)\right]\alpha=J(k)\alpha=0.$$

Since $\alpha\neq 0$ it follows that $J(k)$ is not invertible. Conversely, suppose that $J(k)$ is not invertible for $k\in\mathbb{C}_+$. Then, there exists a nonzero vector $\alpha\in\Omega$ such that $J(k)\alpha=0$. Let $y(k,x):=F(k,x)\alpha$. Then it easily follows that $y(k,x)\in H_1$ and

$$Ay(k,0)+By^{'}(k,0)=\left[AF(k,0)+BF^{'}(k,0)\right]\alpha=J(k)\alpha=0.$$

Therefore, k^2 is an eigenvalue. Q.E.D.

Example 3.2. If $B=0$ and $A=I$ in equation (2.2) then the vertex condition reduces to Dirichlet condition. From the definition of the Jost operator (see equation (2.8)) $J(k)$ is a diagonal operator. In this case, it easily follows that $J(k)$ is not invertible if and only if k^2 is an eigenvalue of Schrödinger operator given by (2.1) on an edge e_i. Therefore, the point spectrum of the infinite star graph Γ is the union of the point spectrums of Schrödinger operators given by (2.1) on individual edges e_i. Moreover, if we assume $q_j(x)$ $(j=1,2,...)$ are real-valued, $Q(x)$ is compact operator in Ω for almost all x, and

$$\int_0^\infty (1+x)\left\|Q(x)\right\|dx<\infty,$$

holds, then Schrödinger operator L defined on H_1 (or equivalently infinite star graph Γ) has a finite negative spectrum (see Theorem 2 in [2]).

Example 3.3. If $B = I$ in equation (2.2), then the vertex condition reduces to Robin condition. If in addition $q_j(x)$ $(j = 1, 2, ...)$ are real-valued, $Q(x)$ is compact operator in Ω for almost all x, the operator A is compact in Ω and

$$\int_0^\infty \|Q(x)\|^2 \, dx < \infty,$$

holds, then it has been shown that Schrödinger operator L defined on H_1 (or equivalently infinite star graph Γ) has a discrete negative spectrum (see Theorem 1 in [2]).

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